XVIII.—*On a Method of Reducing Observations of Underground Temperature, with its Application to the Monthly Mean Temperatures of Underground Thermometers, at the Royal Edinburgh Observatory.* By Joseph D. Everett, M.A., Professor of Mathematics, &c., in King's College, Windsor, N.S., and late Secretary to the Meteorological Society of Scotland.

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A few years since I was engaged in the performance of some calculations under the direction of Professor W. Thomson of Glasgow, having reference to the observations of underground temperature made at the Royal Edinburgh Observatory. In this paper I propose to describe a modification of Professor Thomson's method, which, while retaining a sufficient degree of accuracy, will be simple enough for general adoption. The objects proposed are—

1*st*, To express the variations of temperature at a given depth in terms of the time of year.

2*d*, To deduce the conducting power of the soil.

In the calculations performed for Professor Thomson, the temperatures at 32 equal intervals in each year were required as the basis of calculations; and as the observations had been made only once a-week, it was requisite to interpolate, either by graphical projection (which was the method employed) or in some other way.

A Report of the Royal Edinburgh Observatory having recently passed through my hands, containing the mean temperature of each of the underground thermometers for each month of each year during a period of seventeen years, I have adapted Professor Thomson's method to a computation from 12 (instead of 32) points in the year, and have applied the method thus modified to the means on the seventeen years' observations. The present paper embodies the results, which will be found to agree pretty closely with those obtained by the more elaborate method. The monthly mean temperatures printed in the Observatory Report, on the averages of which, for the seventeen years, the following results are based, are simply the arithmetical means of the weekly readings taken in each calendar month.

For the sake of making the paper intelligible, it will be necessary to premise a few principles which are common to both methods.

The form of expression to which the temperature of each thermometer is to be reduced, is

$$v = A_0 + P_1 \sin\left(2\pi\frac{t}{T} + E_1\right) + P_2 \sin\left(4\pi\frac{t}{T} + E_2\right) + \&c. \quad . \quad . \quad (1.)$$

the general term being $P_n \sin\left(2n\pi\frac{t}{T} + E_n\right)$

Where v is the temperature at the line t from the epoch of reckoning, T is the periodic time (a year), π is the ratio of the circumference of a circle to the diameter, and A_0, P_1, P_2, E_1, E_2, &c. are constants, whose value must be found from the temperatures observed.

It is evident from the form of the expression that A_0 is the mean temperature of the whole year, and that the maximum and minimum values of any subsequent term $P_n \sin\left(2n\pi\frac{t}{T} + E_n\right)$ are $+P_n$ and $-P_n$ respectively. As the range of value through which any term passes depends only on the coefficient P_n, this coefficient is styled the *amplitude* of the term, being in fact equal to half the range.

The epochs of maxima and minima will be very different for different terms. The term involving P_1 has one maximum and one minimum in the year. The term involving P_2 has two maxima and two minima, and generally the term in P_n has n maxima and n minima in the year, its values going through their entire cycle in the $\frac{1}{n}$th part of a year. The term in P_1 is therefore called the annual term, and the term in P_2 the half-yearly term.

The maximum and minimum values of a term will occur earlier or later in the year, according to the value of the constant E_n, any diminution in the value of E_n being the same thing as a retardation of the maxima and minima. Such retardation is called *retardation of phase*. It is the diminution of amplitude and retardation of phase between the terms for thermometers at different depths, that afford the means of deducing the conducting power of the soil.

In order to find from the observed temperatures the values of the constants in expression (1), we must make use of the equivalent expression—

$$v = A_0 + \left(A_1 \cos 2\pi\frac{t}{T} + B_1 \sin 2\pi\frac{t}{T}\right) + \left(A_2 \cos 4\pi\frac{t}{T} + B_2 \sin 4\pi\frac{t}{T}\right) + \&c.$$

the general term being $\left(A_n \cos 2n\pi\frac{t}{T} + B_n \sin 2n\pi\frac{t}{T}\right) \quad . \quad . \quad (2.);$

and then by applying the equations of transformation

$$\sqrt{A_n^2 + B_n^2} = P_n \,,\quad \frac{A_n}{B_n} = \tan E_n \quad . \quad . \quad . \quad . \quad . \quad (3.)$$

we shall obtain the values of the constants in expression (1).

In the calculations performed for Professor THOMSON, the expressions were carried as far as the terms depending on A_4 and B_4. I have carried them only as far as A_2 and B_2, the convergence of the terms being so rapid that a good approximation to the value of the whole series is thus obtained. For deducing the conducting power of the soil even this is more than is required, the values of any single term (except A_0) for the different thermometers being all that theory requires, and the values of A_1 and B_1, inasmuch as they admit of more accurate determination than the coefficients of following terms, can be most advantageously used for making this deduction.

The process for finding the values of A_0, A_1, B_1, &c., is different according to the number of points in the year that are taken. To find analytically the process for 12 points in the year, let

v_0 denote the temperature at the epoch of commencement;

v_1 " " $\frac{1}{12}$ of the year later;

v_2 " " $\frac{2}{12}$ " "

&c. &c.,

and let the sines of 0°, 30°, 60°, and 90°, be denoted by S_0, S_1, S_2, S_3 respectively; we have then the following values of v for the 12 points in the year:—

I.	II.
$v_0=A_0+A_1\,S_3+A_2\,S_3+B_1\,S_0+B_2\,S_0$	$v_6=A_0-A_1\,S_3+A_2\,S_3-B_1\,S_0+B_2\,S_0$
$v_1=A_0+A_1\,S_2+A_2\,S_1+B_1\,S_1+B_2\,S_2$	$v_7=A_0-A_1\,S_2+A_2\,S_1-B_1\,S_1+B_2\,S_2$
$v_2=A_0+A_1\,S_1-A_2\,S_1+B_1\,S_2+B_2\,S_2$	$v_8=A_0-A_1\,S_1-A_2\,S_1-B_1\,S_2+B_2\,S_2$
$v_3=A_0+A_1\,S_0-A_2\,S_3+B_1\,S_3-B_2\,S_0$	$v_9=A_0-A_1\,S_0-A_2\,S_3-B_1\,S_3+B_2\,S_0$
$v_4=A_0-A_1\,S_1-A_2\,S_1+B_1\,S_2-B_2\,S_2$	$v_{10}=A_0+A_1\,S_1-A_2\,S_1-B_1\,S_2-B_2\,S_2$
$v_5=A_0-A_1\,S_2+A_2\,S_1+B_1\,S_1-B_2\,S_2$	$v_{11}=A_0+A_1\,S_2+A_2\,S_1-B_1\,S_1-B_2\,S_2$

Subtracting the quantities in column II. from those in column I. we find

III.	IV.
$v_0-v_6=\ \ 2\,A_1\,S_3+2\,B_1\,S_0$	
$v_1-v_7=\ \ 2\,A_1\,S_2+2\,B_1\,S_1$	$v_5-v_{11}=-2\,A_1\,S_2+2\,B_1\,S_1$
$v_2-v_8=\ \ 2\,A_1\,S_1+2\,B_1\,S_2$	$v_4-v_{10}=-2\,A_1\,S_1+2\,B_1\,S_2$
$v_3-v_9=-2\,A_1\,S_0+2\,B_1\,S_3$	

Subtracting the quantities in column IV. from those opposite to them respectively in column III. (remembering that $S_0=0$, and $S_3=1$), we obtain the remainders—

$2\,A_1 \qquad 4\,A_1\,S_2 \qquad 4\,A_1\,S_1 \qquad 2\,B_1$

If these be multiplied respectively by the factors,

$1 \qquad S_2 \qquad S_1 \qquad 0$

the products are

$2\,A_1 \qquad 4\,A_1\,(S_2)^2 \qquad 4\,A_1\,(S_1)^2$

and the sum of these four products (since $(S_1)^2+(S_2)^2=1$) is $6\ A_1$. Hence the value of A_1 can be found.

Again, adding the quantities which stand opposite to each other in columns III. and IV. we have the sums

$$2\ A_1 \qquad 4\ B_1\ S_1 \qquad 4\ B_1\ S_2 \qquad 2\ B_1;$$

and if we multiply these respectively by the factors,

$$0 \qquad S_1 \qquad S_2 \qquad 1$$

we obtain the products,

$$0 \qquad 4\ B_1\ (S_1)^2 \qquad 4\ B_1\ (S_2)^2 \qquad 2\ B_1$$

The sum of these products is $6\ B_1$; hence B_1 can be found.

Adding the terms opposite each other in columns I. and II. we find

V.	VI.
$v_0+v_6=2\ A_0+2\ A_2\ S_3+2\ B_2\ S_0$	$v_3+v_9=2\ A_0-2\ A_2\ S_3-2\ B_2\ S_0$
$v_1+v_7=2\ A_0+2\ A_2\ S_1+2\ B_2\ S_2$	$v_4+v_{10}=2\ A_0-2\ A_2\ S_1-2\ B_2\ S_2$
$v_2+v_8=2\ A_0-2\ A_2\ S_1+2\ B_2\ S_2$	$v_5+v_{11}=2\ A_0+2\ A_2\ S_1-2\ B_2\ S_2$

The sum of all the terms in V. and VI. is $12\ A_0$, which is in fact the sum of the 12 values of v.

Subtracting the quantities in VI. from those opposite to them in V., we have the remainders,—

$$4\ A_2 \qquad 4\ A_2\ S_1+4\ B_2\ S_2 \qquad -4\ A_2\ S_1+4\ B_2\ S_2.$$

Multiply these remainders respectively by

$$1 \qquad S_1 \qquad -S_1$$

and omitting the two terms $4\ B_2\ S_1\ S_2$ and $-4\ B_2\ S_1\ S_2$, which destroy one another, we have the products—

$$4\ A_2 \qquad 4\ A_2\ (S_1)^2 \qquad 4\ A_2\ (S_1)^2$$

whose sum (since $S_1=\frac{1}{2}$) is $6\ A_2$. Hence A_2 can be found.

Again, if the above remainders be multiplied respectively by

$$0 \qquad S_2 \qquad S_2$$

the products (omitting terms which destroy each other) are

$$0 \qquad 4\ B_2\ (S_2)^2 \qquad 4\ B_2\ (S_2)^2$$

and since $S_2=\frac{\sqrt{3}}{2}$, the sum of these products is $6\ B_2$. Hence B_2 can be found.

The application of the process above indicated to the determination of A_1 B_1 A_2 B_2 for the 3 feet thermometer is subjoined.

I. Temperatures of first 6 months.	II. Temperatures of last 6 months.	III. I. − II.	IV. Last two Nos. in III. reversed.	III. − IV.	Multipliers.	Products.	III. + IV.	Multipliers.	Products.
40·57	52·70	−12·13		−12·13	1	−12·13	−12·13	0	− 0·00
39·64	53·82	−14·18	+7·24	−21·42	S_2	−18·55	− 6·94	S_1	− 3·47
40·31	52·75	−12·44	+0·35	−12·79	S_1	− 6·39	−1209	S_2	−10·47
42·45	49·15	− 6·70		− 6·70	0	·00	− 6·70	1	− 6·70
45·87	45·52	+ 0·35			6)	−37·07		6)	−20·64
49·86	42·62	+ 7·24			$A_1=$	− 6·18		$B_1=$	− 3·44

V. First Half of (I.+II.)	VI. Last Half of (I.+II.)	V. − VI.	Multipliers.	Products.	V. − VI. again.	Multipliers.	Products.
93·27	91·60	+1·67	1	+1·67	1·67	0	·00
93·46	91·39	+2·07	S_1	+1·035	+2·07	S_2	+2·295
93·06	92·48	+0·58	$-S_1$	−0·290	+0·58	S_2	
			6)	+2·415		6)	+2·295
			$A_2=$	+ ·4025		$B_2=$	+ ·3825

A_0 = mean of all the numbers in I. and II. = 46·27.

There are in all four thermometers, their bulbs being sunk to depths of 3, 6, 12, and 24 French feet respectively below the surface of the ground. The means of their readings, in degrees Fahrenheit, for each calendar month, on the average of the seventeen years 1838–1854, are as under:—

Depth of Thermometer.	Jan.	Feb.	Mar.	April.	May.	June.	July.	Aug.	Sept.	Oct.	Nov.	Dec.
3 feet	40·57	39·64	40·31	42·45	45·87	49·86	52·70	53·82	52·75	49·15	45·52	42·62
6 feet	43·59	42·35	42·00	42·79	44·65	47·23	49·71	51·31	51·54	50·11	47·81	45·48
12 feet	46·84	45·82	45·06	44·68	44·88	45·63	46·84	48·07	48·96	49·27	49·02	47·94
24 feet	47·77	47·63	47·39	47·08	46·79	46·59	46·55	46·69	46·97	47·31	47·61	47·79

The values of A_0, A_1, B_1, A_2, B_2, obtained in the manner above indicated, are—

	A_0	A_1	B_1	A_2	B_2
For the 3 feet thermometer,	46·27	−6·18	−3·44	+·4025	+·3825
... 6 feet ...	46·55	−3·10	−3·65	+·120	+·293
... 12 feet ...	46·92	+0·03	−2·31	−·0833	+·0635
... 24 feet ...	47·18	+0·615	−0·118	−·0167	−·0144

And the values of P_1, P_2, E_1, E_2, obtained from these by the formulæ of transformation (3), are—

	P_1	P_2	E_1	E_2
For the 3 feet thermometer,	7·07	·56	240° 54′	46° 27½′
... 6 feet ...	4·79	·32	220° 20′	22° 16′
... 12 feet ...	2·31	·10	179° 15′	− 52° 41′
... 24 feet ...	·63	·02	100° 52′	−130° 46′

With the view of testing how nearly the formulæ give the true temperature of each month, I have calculated the temperature of each thermometer for each month both by formula (1) and formula (2), the results in the two cases being identical; and the following table exhibits their differences from the actual temperatures. The numbers in the first line are the actual temperatures; those in the second line are obtained by putting t successively equal to 0, $\frac{1}{12}$ T, $\frac{2}{12}$ T, &c. in expressions (1) and (2); and those in the third line are the corrections necessary for reducing the calculated to the actual temperatures. For the sake of exhibiting the variations of temperature more clearly, the temperatures have in each case been diminished by the mean of the year, so that temperatures below the mean bear the negative sign.

	Jan.	Feb.	Mar.	April.	May.	June.	July.	Aug.	Sept.	Oct.	Nov.	Dec.
3 feet Ther.												
Actual,	−5·70	−6·63	−5·96	−3·82	−0·40	+3·59	+6·43	+7·55	+6·48	+2·88	−0·75	−3·65
By formula,	−5·78	−6·54	−5·94	−3·84	−0·42	+3·50	+6·58	+7·60	+6·20	+3·04	−0·64	−3·76
Difference,	+ ·08	− ·09	− ·02	+ ·02	+ ·02	+ ·09	− ·15	− ·05	+ ·28	− ·16	− ·11	+ ·11
6 feet Ther.												
Actual,	−2·96	−4·20	−4·55	−3·76	−1·90	+0·68	+3·16	+4·76	+4·99	+3·56	+1·26	−1·07
By formula,	−2·98	−4·20	−4·52	−3·77	−1·93	+0·67	+3·22	+4·82	+4·91	+3·53	+1·30	−1·05
Difference,	+ ·02	·00	− ·03	+ ·01	+ ·03	+ ·01	− ·06	− ·06	+ ·08	+ ·03	− ·04	− ·02
12 feet Ther.												
Actual,	−0·08	−1·10	−1·86	−2·24	−2·04	−1·29	−0·08	+1·15	+2·04	+2·35	+2·10	+1·02
By formula,	−0·05	−1·12	−1·89	−2·23	−2·03	−1·28	−0·11	+1·14	+2·08	+2·39	+2·00	+1·08
Difference,	− ·03	+ ·02	+ ·03	− ·01	− ·01	− ·01	+ ·03	+ ·01	− ·04	− ·04	+ ·10	− ·06
24 feet Ther.												
Actual,	+ ·59	+ ·45	+ ·21	− ·10	− ·39	− ·59	− ·63	− ·49	− ·21	+ ·13	+ ·43	+ ·61
By formula,	+ ·60	+ ·45	+ ·20	− ·10	− ·39	− ·59	− ·63	− ·49	− ·21	+ ·13	+ ·43	+ ·60
Difference,	− ·01	·00	+ ·01	·00	·00	·00	·00	·00	·00	·00	·00	+ ·01

If we had neglected the term involving P_2 (the half-yearly term), and taken only the term involving P_1 (the annual term), we should have obtained the results entered in the first line of the following table. The numbers in the second line are the corrections necessary for reducing these results to the actual temperatures.

	Jan.	Feb.	Mar.	April.	May.	June.	July.	Aug.	Sept.	Oct.	Nov.	Dec.
Results,	−6·18	−7·07	−6·07	−3·44	+0·11	+3·63	+6·18	+7·07	+6·07	+3·44	−0·11	−3·63
Corrections,	+ ·48	+ ·44	+ ·11	− ·38	− ·51	− ·04	·25	+ ·48	+ ·41	− ·56	− ·64	− ·02
Results,	−3·10	−4·51	−4·71	−3·65	−1·61	+0·86	+3·10	+4·51	+4·71	+3·65	+1·61	−0·86
Corrections,	+ ·14	+ ·31	+ ·16	− ·11	− ·29	− ·18	+ ·06	+ ·25	+ ·28	− ·09	− ·35	− ·21
Results,	+0·03	−1·13	−1·98	−2·31	−2·02	−1·18	−0·03	+1·13	+1·99	+2·31	+2·02	+1·18
Corrections,	− ·11	+ ·03	+ ·12	+ ·07	− ·02	− ·11	− ·05	+ ·02	+ ·05	+ ·04	+ ·08	− ·16
Results,	+ ·62	+ ·47	+ ·20	− ·12	− ·41	− ·59	− ·62	− ·47	− ·21	+ ·12	+ ·41	+ ·59
Corrections,	− ·03	− ·02	+ ·01	+ ·02	+ ·02	·00	− ·01	− ·02	·00	+ ·01	+ ·02	+ ·02

The processes hitherto described are applicable not only to underground temperatures, but also to open-air temperatures, and, in fact, to any element that varies in a regular manner.

It remains to show how the results which we have obtained can be applied for determining the conductivity of the soil. The mode of procedure will be exactly the same as that adopted in the calculations for Professor THOMSON.

The conducting power of the soil may be inferred either from the diminution in the values of P_1 and P_2 as we descend in the soil, or from the diminution of E_1 and E_2. In other words, it may be inferred either from diminution of amplitude, or from retardation of phase.

Let x denote the difference in depth of any two of the thermometers; let $\Delta . E_n$ denote the retardation of phase, or the excess of the value of E_n, for the upper of the two thermometers above its value for the lower, E_n being expressed not in degrees and minutes, but in circular measure; and let $\Delta . \log_e P_n$ denote the diminution of the Napierian logarithm of the amplitude, or the excess of $\log_e P_n$ for the upper thermometer above $\log_e P_n$ for the lower; the ratio of k, the conductivity of the soil, to c, the capacity of the soil, for heat, may then be determined by either of the equations

$$\frac{\Delta . E_n}{x}=\sqrt{\frac{n\pi c}{Tk}} \qquad \frac{\Delta . \log_e P_n}{x}=\sqrt{\frac{n\pi c}{Tk}} \quad . \quad . \quad . \quad . \quad (4)$$

The manner in which these equations are deduced from the differential equation for the flow of heat through the soil,

$$\frac{dv}{dt}=\frac{k}{c} \cdot \frac{d^2v}{dx^2},$$

will be stated in a note at the end of this paper. At present we proceed to apply the equations to the numerical results above obtained.

The values of E_1 and E_2 in circular measure, and of $\log_e P_1$ and $\log_e P_2$, are as under:—

	$\text{Log}_e\ P_1$	$\text{Log}_e\ P_2$	E_1 in circular measure.	E_2 in circular measure.
3 feet thermometer	1·95	− ·59	4·20	+ ·81
6 feet "	1·56	−1·15	3·85	+ ·39
12 feet "	·84	−2·25	3·15	− ·92
24 feet "	− ·47	−3·81	1·76	−2·28

By comparing the thermometers two and two in every possible combination, the following results are obtained:—

FOR THE ANNUAL TERM.

Thermometers compared.	ΔE_1	x	$\sqrt{\frac{\pi c}{Tk}}$	$\Delta \log_e P_1$	x	$\sqrt{\frac{\pi c}{Tk}}$
3 feet and 6 feet.	·35	3	·117	.39	3	·130
3 feet and 12 feet.	1·05	9	·117	1·11	9	·123
3 feet and 24 feet.	2·44	21	·116	2·42	21	·115
6 feet and 12 feet.	·70	6	·117	·72	6	·120
6 feet and 24 feet.	2·09	18	·116	2·03	18	·113
12 feet and 24 feet.	1·39	12	·116	1·31	12	·109
Means.			·1165	. . . .		·1183

FOR THE HALF-YEARLY TERM.

Thermometers compared.	ΔE_2	x	$\sqrt{\frac{2\pi c}{Tk}}$	$\Delta \log_e P_2$	x	$\sqrt{\frac{2\pi c}{Tk}}$
3 feet and 6 feet.	·42	3	·140	·56	3	·187
3 feet and 12 feet.	1·73	9	·192	1·66	9	·184
3 feet and 24 feet.	3·09	21	·147	3·22	21	·153
6 feet and 12 feet.	1·31	6	·218	1·10	6	·183
6 feet and 24 feet.	2·67	18	·148	2·66	18	·148
12 feet and 24 feet.	1·36	12	·113	1·56	12	·130
Means, ...			·160	. . . .		·164
Quotients by $\sqrt{2}$, ...			·113	. . . .		·116

The results deduced from the annual term agree the best among themselves, and are the most reliable; the coefficients P_2 of the half-yearly term being very small, and varying considerably from year to year. Notwithstanding, the mean values ·113, ·116 of $\sqrt{\frac{\pi c}{Tk}}$, deduced from the half-yearly term, agree very well with the values ·1165, ·1183 from the annual term. Professor THOMSON's results from the temperatures of the thirteen years 1842–1854 were:—

	By Phase.	By Amplitude.
For the annual term, . . .	·1156	·1160
For the half-yearly term, . .	·08861	·11133

—these numbers being the values of the function $\sqrt{\frac{\pi c}{Tk}}$ obtained from the coefficients in the same manner as above. The agreement as regards the annual term is very remarkable, extending, as it does, both in the determination from phase and in that from amplitude to the fourth decimal place.

Note on the Equations

$$\frac{\Delta \,.\, \mathrm{E}}{x}=\sqrt{\frac{n\pi c}{\mathrm{T}k}}=\frac{\Delta \,.\, \log_e \mathrm{P}_n}{x}.$$

The differential equation for the conduction of heat through the soil, the surface being supposed horizontal and the soil uniform, is

$$\frac{dv}{dt}=\frac{k}{c}\cdot\frac{d^2v}{dx^2}.$$

This equation is satisfied if we assume

$$v=\mathrm{P}e^{-x\sqrt{\frac{n\pi c}{\mathrm{T}k}}}\sin\left(2n\pi\frac{t}{\mathrm{T}}+\mathrm{E}_n-x\sqrt{\frac{n\pi c}{\mathrm{T}k}}\right)$$

—e being the base of Napierian logarithms, and P any constant.

To show that this integral satisfies the differential equation, put

$$\sqrt{\frac{n\pi c}{\mathrm{T}k}}=\alpha, \quad \frac{2n\pi t}{\mathrm{T}}+\mathrm{E}-\beta.$$

The equation then becomes

$$v=\mathrm{P}e^{-\alpha x}\sin(\beta-\alpha x).$$

Whence

$$\frac{dv}{dt}=\frac{d\beta}{dt}\,.\,\mathrm{P}e^{-\alpha x}\cos(\beta-\alpha x)=\frac{2n\pi}{\mathrm{T}}\,.\,\mathrm{P}e^{-\alpha x}\cos(\beta-\alpha x)$$

$$\frac{dv}{dx}=-\mathrm{P}\alpha e^{-\alpha x}\left\{\sin(\beta-\alpha x)+\cos(\beta-\alpha x)\right\}$$

$$\frac{d^2v}{dx^2}=\mathrm{P}\alpha^2e^{-\alpha x}\left\{\sin(\beta-\alpha x)+\cos(\beta-\alpha x)+\cos(\beta-\alpha x)-\sin(\beta-\alpha x)\right\}$$

$$=2\,\mathrm{P}\alpha^2e^{-\alpha x}\cos(\beta-\alpha x).$$

Hence

$$\frac{dv}{dt}=\frac{n\pi}{\mathrm{T}}\cdot\frac{1}{\alpha^2}\cdot\frac{d^2v}{dx^2};\ \text{but}\ \frac{1}{\alpha^2}=\frac{\mathrm{T}k}{n\pi c}.$$

Whence

$$\frac{dv}{dt}=\frac{k}{c}\cdot\frac{d^2v}{dx^2},$$ or the differential equation is satisfied.

It will be equally satisfied if, instead of a single term, we have a series of terms of the same form as that above assigned to v, and if we likewise prefix a constant A_0. Hence we have the general equation

$$v=A_0+P_1e^{-x\sqrt{\frac{\pi c}{Tk}}}\sin\left(2\pi\frac{t}{T}+E_1-x\sqrt{\frac{\pi c}{Tk}}\right)+P_2e^{-x\sqrt{\frac{2\pi c}{Tk}}}\sin\left(4\pi\frac{t}{T}E_2-x\sqrt{\frac{2\pi c}{Tk}}\right)+\&c.$$

the general term being

$$P_ne^{-x\sqrt{\frac{n\pi}{Tk}}}\sin\left(2n\pi\frac{t}{T}+E_n-x\sqrt{\frac{n\pi c}{Tk}}\right).$$

In this equation x denotes the distance below any assumed horizontal plane. Let the plane pass through the bulb of one of the thermometers; then the general term will become for this thermometer

$$P^n\sin\left(2n\pi\frac{t}{T}+E^n\right);$$

while for a thermometer lower by x feet it is

$$P\,e^{-x\sqrt{\frac{n\pi}{Tk}}}\sin\left(2n\pi\frac{t}{T}+E_n-x\sqrt{\frac{n\pi c}{Tk}}\right).$$

Hence it appears that in descending through x feet the amplitude P_n is diminished in the ratio of $e^{-x\sqrt{\frac{n\pi c}{Tk}}}$ to 1, while the quantity E_n is diminished by the amount $x\sqrt{\frac{n\pi c}{Tk}}$. Whence the equations

$$\frac{\Delta\,.\,\log_e P}{x}=\frac{\Delta\,E}{x}=\sqrt{\frac{n\pi c}{Tk}}.$$